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WREATH PRODUCTS AND P.I. ALGEBRAS

Antonino GIAMBRUNO
Universita di Palermo, Istituto di Matematica, Via Archirafi 34, I-90123 Palermo, Italy

Amitai REGEV*
Math. Dept., Pennsylvania State University, University Park, PA 16802, USA

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The representation theory of wreath products $G \sim S_n$ is applied to study algebras satisfying polynomial identities that involve a group G of (anti)automorphisms, in the same way the representation theory of S_n was applied earlier to study ordinary P.I. algebras. Some of the basic results of the ordinary case are generalized to the G-case.

0. Introduction

Throughout this paper we assume F is a field of characteristic zero, and all algebras considered here are F-algebras.

The representation theory of the symmetric group S_n has proved to be a very useful tool in the study of P.I. algebras [2], [4], [12], [13], [15], etc. The basic idea here is to identify the space $V_n(x)$, of the multilinear polynomials in $x_1, \ldots, x_n$, with the group algebra $FS_n : V_n(x) \equiv FS_n$. If $Q = I(R)$ are the (ordinary) identities of R, this makes $Q_n = Q \cap V_n$ a left ideal in FS_n, and allows us to define the sequences of cocharacters $\chi_n(R)$ and codimensions $c_n(R)$ [2], [11], [12], etc.

Let R be an F-algebra and let G be a group of automorphisms and anti-automorphisms of R. G-polynomials and G-polynomial identities (G-P.I.) are defined in a natural way [7], [9]. An important class of such algebras are rings with involution $*$ [1], [5], [9]; $*$-polynomial identies where characterized by Amitsur [1], who showed that a ring with involution $*$ is P.I. iff it is $*$-P.I.

Let G be a group, $G \sim S_n$ its wreath product with S_n [6], and let R be a G-P.I. algebra. In this paper we show how the representation theory of $G \sim S_n$ can be applied to the study of the G-identities of R. This is done in a way which generalizes the ordinary case – in which the representation theory of S_n is applied to P.I. algebras. Here we (again!) identify the group algebra $F[G \sim S_n]$ with $V_n(x \mid G)$, the multilinear G-polynomials of degree n: if $P = G.I(R)$ are the G-polynomial identi-

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ties of R, then $P_n = P \cap V_n(x \mid G)$ is a left ideal in $F[G \sim S_n]$; the G-cocharacters $\chi_n(R \mid G)$ are defined accordingly.

The applications of $G \sim S_n$ representations require a detailed knowledge of the idempotents and the one-sided ideals in $F[G \sim S_n]$. A detailed representation theory of $\mathbb{Z}_2 \sim S_n$ was obtained by A. Young [17]. The general method for obtaining the irreducible representations of wreath products over $\mathbb{C}$ was later obtained by Specht [8], [16]. In [14], these representations are obtained from a double centralizing theorem. In the Appendix here we derive, from [14], a detailed and explicit information about idempotents, one-sided ideals and 'Branching' in $F[\mathbb{Z}_1 \sim S_n]$; this is essential for the applications of $\mathbb{Z}_2 \sim S_n$ representations to rings with involution. The few basic properties of the (ordinary) identification $V_n(x) \equiv FS_n$ are reproved here, in Section 2, in the G-case, thus allowing us later to generalize some of the 'ordinary' results. Such are the characterizations of Capelli identities [13], and 'hook' properties for the cocharacters [2]; they are redone here (Section 5) in the case of rings with involution – and could be done in a more general situation (to shorten and to make the exposition explicit we do not treat the subject in the most general possible way!).

We finally deduce some initial results about the $*$-characters of the $k \times k$ matrices F_k (A^* being the transpose of $A \in F_k$). These simple results hint that a single Young diagram (partition) $\theta \vdash n$ in the ordinary cocharacter $\chi_n(F_k)$ is replaced in $\chi_n(F_k \mid *)$, somehow, by a set of pairs of partitions (λ, μ) with $\approx$ half the height of θ. It is hoped that a further study will yield some interesting results about both the ordinary and the $*$-cocharacters of F_k.

1. Wreath products

Let A be a vector space and write

$$T^n(A) = \underbrace{A \otimes \cdots \otimes A}_{n \text{ times}}.$$

The symmetric group S_n acts on $T^n(A)$ by permuting coordinates:

$$\sigma \in S_n, a = a_1 \otimes \cdots \otimes a_n \in T^n(A), \text{ then } \sigma(a) = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}.$$

In the case A is an algebra, S_n clearly acts on $T^n(A)$ as a group of automorphisms, and we define the wreath-product $A \sim S_n$ to be the twisted group algebra $T^n(A)\langle S_n \rangle : T^n(A)\langle S_n \rangle \overset{\text{def}}{=} T^n(A) \otimes F[S_n]$ as vector spaces, and multiplication is given by

$$(a \otimes \sigma) \cdot (b \otimes \tau) \overset{\text{def}}{=} a \cdot \sigma(b) \otimes \sigma t, \quad a, b \in T^n(A), \sigma, \tau \in S_n \qquad [8].$$

If G is any group, the wreath product $G \sim S_n$ (which is a group!) is defined similarly [6], and one easily verifies that $F[G \sim S_n] = (F[G]) \sim S_n$.

For the representation theory of wreath products, see the introduction.

2. Identifying $F[G \sim S_n] \equiv V_n(x \mid G)$

2.1. Let G be a group, X a set of indeterminates, then form the (larger) set of indeterminates

$$X \times G \equiv \langle X \mid G \rangle = \{x^g = g(x) \mid x \in X, g \in G\}.$$

G acts naturally on $\langle X \mid G \rangle$:

$$(x^{g_1})^{g_2} = x^{(g_2 g_1)} \quad \text{(i.e. } g_2(g_1(x)) = (g_2 g_1)x) \quad \text{for } x \in X, g_1, g_2 \in G.$$

We let $F\langle X \mid G \rangle$ be the (associative) ring of non-commutative F-polynomials in the indeterminates $\langle X \mid G \rangle$. The difference between these and the ordinary case (no- or trivial-G) is in the degree function:

2.2. Definition. Let $M \in F\langle X \mid G \rangle$ be a monomial and let $y \in X$. Then the degree of M in y, $\deg_y M$, is defined as the number of times the variables y^g appear in M (disregarding the g's $\in (G)$.

2.3. Definition. Let $X, G, F\langle X \mid G \rangle, \deg_y M$ as in 2.1, 2.2. Assume now that $|X| = \infty$ and fix a sequence $x_1, x_2, \ldots \in X$.

We define the space of G-multilinear polynomials $V_n(x_1, \ldots, x_n \mid G) = V_n(x \mid G)$ as follows:

$$V_n(x \mid G) = \operatorname{span}_F \{x_{\sigma(1)}^{g_1} \cdots x_{\sigma(n)}^{g_n} \mid \sigma \in S_n, g_i \in G\}.$$

We now identify $V_n(x \mid G)$ with $F[G \sim S_n]$ in a way that generalizes the identification $V_n(x) \equiv FS_n$ (G-trivial) [11]. This is done in

2.4. Definition. Let $G^{(n)} = \underbrace{G \times \cdots \times G}_{n}$, so $F[G^{(n)}] \equiv T^n(F[G])$. Let

$$g = (g_1, \ldots, g_n) \equiv g_1 \otimes \cdots \otimes g_n \in G^{(n)}, \qquad \sigma \in S_n,$$

so $g \otimes \sigma \in G \sim S_n$. Then identify

$$g \otimes \sigma \equiv M_{g \otimes \sigma}(x_1, \ldots, x_n) \stackrel{\text{def}}{=} x_{\sigma(1)}^{g_{\sigma(1)}^{-1}} \cdots x_{\sigma(n)}^{g_{\sigma(n)}^{-1}}.$$

Extend, by linearity, to identify

$$F[G \sim S_n] \equiv V_n(x \mid G).$$

Note. The identification $V_n(x) \equiv FS_n$ (G trivial) has two basic properties [12, §2, (1), (2)] which made it possible to apply the theory of S_n-representations to P.I. algebras. Fortunately, these two properties (easily) extend to the identification $F[G \sim S_n] \equiv V_n(x \mid G)$:

2.5. Lemma. (1) *Let* $g, h \in G^{(n)}, \sigma, \tau \in S_n$. *Then*

$$(h \otimes \tau) \cdot M_{g\otimes\sigma}(x_1, \ldots, x_n) = M_{g\otimes\sigma}\left(x_{\tau(1)}^{h_{\tau(1)}^{-1}}, \ldots, x_{\tau(n)}^{h_{\tau(n)}^{-1}}\right).$$

(2) *Let* $\eta \in S_n \subseteq G \sim S_n$, *and write* $M_{g\otimes\sigma}(x_1, \ldots, x_n) = y_1 \cdots y_n$. *Then*

$$M_{g\otimes\sigma}(x_1, \ldots, x_n)\eta = y_{\eta(1)} \cdots y_{\eta(n)},$$

i.e., multiplication by a permutation from the right changes the order (*places*) *in every monomial by that permutation.*

Proof. We prove, for example, (1)

$$(h \otimes \tau) \cdot M_{g\otimes\sigma}(x) \equiv (h \otimes \tau)(g \otimes \sigma) = h \cdot \tau(g) \otimes \tau\sigma = k \otimes \theta$$

where $\theta = \tau\sigma$ and $k_i = h_i \cdot g_{\tau^{-1}(i)}$, $1 \le i \le n$. Now

$$k \otimes \theta \equiv x_{\theta(1)}^{k_{\theta(1)}^{-1}} \cdots x_{\theta(n)}^{k_{\theta(n)}^{-1}} \quad \text{and} \quad k_{\theta(j)}^{-1} = (h_{\theta(j)} g_{\tau^{-1}\theta(j)})^{-1} = g_{\sigma(j)}^{-1} \cdot h_{\tau\sigma(j)}^{-1}.$$

Hence $x_{\theta(j)}^{k_{\theta(j)}^{-1}} = \left(x_{\tau\sigma(j)}^{h_{\tau\sigma(j)}^{-1}}\right)^{g_{\sigma(j)}^{-1}}$ (see 2.1) which implies

$$k \otimes \theta = \left(x_{\tau\sigma(1)}^{h_{\tau\sigma(1)}^{-1}}\right)^{g_{\sigma(1)}^{-1}} \cdots \left(x_{\tau\sigma(n)}^{h_{\tau\sigma(n)}^{-1}}\right)^{g_{\sigma(n)}^{-1}} = M_{g\otimes\sigma}\left(x_{\tau(1)}^{h_{\tau(1)}^{-1}}, \ldots, x_{\tau(n)}^{h_{\tau(n)}^{-1}}\right)$$

(2) The proof of (2) is similar. □

3. G-T-ideals

3.1. Notations. Let R be an F algebra, and let $\mathrm{Aut}^*(R)$ denote the group of all automorphisms and anti-automorphisms of R. The subgroup $\mathrm{Aut}(R)$ of R automorphisms is normal, of index ≤ 2, in $\mathrm{Aut}^*(R)$. Let $G \subseteq \mathrm{Aut}^*(R)$. Given $f(x_1, \ldots, x_m) \in F\langle X \mid G\rangle$ and $r_1, \ldots, r_n \in R$, one evaluates $f(r_1, \ldots, r_n) \in R$; if $f(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$, $f(x)$ is a G-identity and R is a G-P.I. algebra.

Denote $P = G.I(R) = \{$the G-identities of $R\} \subseteq F\langle X \mid G\rangle$. Then P is a G-T-ideal in the sense of 3.3. We first make G act on $F\langle X \mid G\rangle$:

3.2. Definition. Let G be a group, $H \trianglelefteq G$ a normal subgroup (interpret H as automorphisms, $G \setminus H$ as anti-automorphisms. For example, if $G \subseteq \mathrm{Aut}^*(R)$, then $H = G \cap \mathrm{Aut}(R)$). As in 2.1, G acts on $\langle X \mid G\rangle$. Extend to $F\langle X \mid G\rangle = F\langle X \mid H \trianglelefteq G\rangle$:

Let M, N be monomials, $g \in G$, then

$$(MN)^g = \begin{cases} M^g \cdot N^g & \text{if } g \in H, \\ N^g \cdot M^g & \text{if } g \in G \setminus H. \end{cases}$$

By linearity, G now acts on $F\langle X \mid G\rangle$ with H as automorphisms, $G - H$ as anti-automorphisms.

3.3. Definition. (a) Let $H \trianglelefteq G$ act on $F\langle X \mid G\rangle$ as in 3.2. Then $\varphi : F\langle X \mid G\rangle \to F\langle X \mid G\rangle$ is a G-homomorphism if for all $x \in X$ and $g \in G$, $\varphi(x^g) = (\varphi(x))^g$.

(b) The two-sided ideal $P \subseteq F\langle X \mid G\rangle$ is a G-T-ideal if for all such G-homomorphisms φ, $\varphi(P) \subseteq P$.

3.4. Corollary. *The G-identities P of a G-P.I algebra R* (3.1) *is a G-T-ideal in $F\langle X \mid G\rangle$.*

4. G-Codimensions and G-cocharacters

4.1. Corollary. *Let $P \subseteq F\langle X \mid G\rangle$ be a G-T-ideal. It easily follows from* 2.4(1) *that $P_n = P \cap V_n(x \mid G)$ is a left-ideal in $F[G \sim S_n] \equiv V_n(x \mid G)$, so $V_n(x \mid G)/P_n$ is a left $F[G \sim S_n]$ module.*

4.2. Definitions. Let R be an F algebra, $G \subseteq \mathrm{Aut}^*(R)$ a subgroup, $H = G \cap \mathrm{Aut}(R)$ and let $P \subseteq F\langle X \mid G\rangle$ be the G-identities of R. Following the case when G is trivial [11], we now define $\chi_n(R \mid G)$ to be the $G \sim S_n$ character of the module $V_n(x \mid G)/P_n$; we call $\{\chi_n(R \mid G)\}$ 'the G-cocharacters of R'. The 'G-codimensions' of R are $c_n(R \mid G) \stackrel{\text{def}}{=} \dim(V_n(x \mid G)/P_n)$ and are the degrees of the G-cocharacters.

4.3. Remark. Given $R, G \subseteq \mathrm{Aut}^*(R)$ as in 4.2, we can also ignore G: we have the ordinary polynomials $F\langle X\rangle$, the polynomial identities $Q = I(R) \subseteq F\langle X\rangle$ and the space $V_n(x)$ of multilinear polynomials in $x_1, \dots, x_n$. Thus $Q_n = Q \cap V_n$ is a left ideal in FS_n, $\chi_n(R)$ is the character of V_n/Q_n, and $c_n(R) = \dim(V_n/Q_n)$ the ordinary codimensions [11]. We have the following trivial lemma.

4.4. Lemma. *With notations as in* 4.2 *and* 4.3, $c_n(R) \le c_n(R \mid G)$.

Proof. By definition, $c_n(R)$ is the maximal number of monomials in V_n which are linearly independent modulo $Q = I(R)$. Since

$$Q = I(R) = F\langle x\rangle \cap P \qquad (P = G.I(R) \le F\langle X \mid G\rangle),$$

such monomials are also linearly independent modulo P. $\square$

4.5. Example. Let R be a ring with an involution [5, p.17] and denote the identity map by $1 : R \to R$. We have $G = \{1, *\} \subseteq \mathrm{Aut}^*(R)$ and $G \cong \mathbb{Z}_2$. Thus, the representation theory of $\mathbb{Z}_2 \sim S_n$ is applied to study the $*$-polynomial identities of rings with involution. A major example for such rings (algebras) are $k \times k$ matrices over the field F, were $* = \mathrm{T}$ is the transpose.

4.6. Remark. Clearly, if R is P.I. then, for any $G \subseteq \mathrm{Aut}^*(R)$, R is also G-P.I. The converse, in general, is not true: a counterexample was given by Kharchenco

[9, p.103]. However, by Amitsur's theorem [1], that converse is true for rings with involutions: $*$-P.I. implies P.I.! The following theorem translates the question of whether G-P.I. implies P.I. into the language of codimensions.

It is known that an algebra R is P.I. iff $c_n(R)$ is exponentially bounded [10], [12, Theorem 1.1].

4.7. Lemma. *Let R be G-P.I. and (ordinary) P.I. satisfying an ordinary identity of degree d. Then*

$$c_n(R \mid G) \leq |G|^n (d-1)^{2n}.$$

Proof. Let $\Omega_n \subseteq S_n$ be a basis (of monomials) for $V_n(x) \equiv FS_n$ modulo the ordinary identities $I(R) = Q$: For all $\sigma \in S_n$

$$x_{\sigma(1)} \cdots x_{\sigma(n)} = M_\sigma(x_1, \ldots, x_n) = \sum_{\tau \in \Omega_n} \alpha(\sigma, \tau) \cdot M_\tau(x_1, \ldots, x_n).$$

Let $1 \otimes \sigma = 1 \otimes \cdots \otimes 1 \otimes \sigma$ $(1 = 1_G)$. Then $M_{1 \otimes \sigma}(x) = M_\sigma(x)$ (2.4). Thus, for any $g \otimes \sigma \in G \sim S_n$,

$$\begin{aligned} M_{g \otimes \sigma}(x) &\equiv g \otimes \sigma = (g \otimes 1)(1 \otimes \sigma) \equiv (g \otimes 1) M_{1 \otimes \sigma}(x) \\ &\overset{2.4(1)}{=} M_{1 \otimes \sigma}\left(x_1^{g_1^{-1}}, \ldots, x_n^{g_n^{-1}}\right) = M_\sigma\left(x_1^{g_1^{-1}}, \ldots, x_n^{g_n^{-1}}\right). \end{aligned}$$

Hence

$$\begin{aligned} M_{g \otimes \sigma}(x) &= \sum_{\tau \in \Omega_n} \alpha(\sigma, \tau) M_\tau\left(x_1^{g_1^{-1}}, \ldots, x_n^{g_n^{-1}}\right) \pmod{I(R)} \\ &= \sum_{\tau \in \Omega_n} \alpha(\sigma, \tau) M_{g \otimes \tau}(x_1, \ldots, x_n) \pmod{I(R)}. \end{aligned}$$

Since $I(R) \subseteq G.I(R)$, this shows

$$c_n(R \mid G) \leq |G|^n c_n(R),$$

and the proof follows from [12, 1.1]. $\square$

As a corollary we have

4.8. Theorem. *Let $G \subseteq \mathrm{Aut}^*(R)$ be a finite subgroup, and let R be a G-P.I. algebra. Then R satisfies an ordinary identity iff $c_n(R \mid G)$ is exponentially bounded (i.e. there exists $0 < a$ such that for all n, $c_n(R \mid G) \leq a^n$).*

We have thus 'translated' Amitsur's theorem to the language of codimensions:

4.9. Amitsur's theorem [1]. *A ring R with involution $*$ that is $*$-P.I. is also (ordinary) P.I.*

Equivalently, such R is $*$-P.I. iff $c_n(R \mid *)$ is exponentially bounded.

Thus, a direct, 'combinatorial' proof of the fact – yet to be founded – would yield another, combinatorial, proof of that theorem.

Similar remarks apply to the other known cases where G-P.I. implies P.I. [9, 6.15].

5. Involutions and $F[\mathbb{Z}_2 \sim S_n] \equiv V_n(x \mid *)$

We now realize some of the idempotents of $F[\mathbb{Z}_2 \sim S_n]$ as $*$-polynomials for rings with involution. We assume the reader is familiar with the appendix.

5.1. Notations. As in the appendix, $n = m_1 + m_2$, $\lambda \vdash m_1$, $\mu \vdash m_2$, $\lambda \leftrightarrow e_\lambda$, $\mu \leftrightarrow e_\mu$, $f(m_1, m_2) = T^{m_1}(f_1) \otimes T^{m_2}(f_2)$ and $e_{\lambda,\mu} = f(m_1, m_2) \otimes (e_\lambda \otimes e_\mu)$.

Following [12, §2] we now realize $e_{\lambda,\mu} = e_{\lambda,\mu}(x_1, \dots, x_n)$, as a $*$-polynomial. As in [12, §2], we begin with the tableau $T_0(\lambda) \leftrightarrow e_{0,\lambda}$ (and $T_0(\mu) \leftrightarrow e_{0,\mu}$). Thus $e_{0,\lambda} \otimes e_{0,\mu} \equiv e_{0,\lambda}(x_1, \dots, x_{m_1}) \cdot e_{0,\mu}(x_{m_1+1}, \dots, x_n)$ is given in [12, §2], and we calculate $e_{0,\lambda,\mu}(x)$.

5.2. Note. $G = \{1, *\} \cong \mathbb{Z}_2$ (4.5), so $f_1 = 1 + *$, $f_2 = 1 - *$ (A.1, with $* = g$): $x^{f_1} = x + x^*$, $x^{f_2} = x - x^*$.

Now, $e_{\lambda,\mu} = [f(m_1, m_2) \otimes 1] \cdot [1 \otimes (e_\lambda \otimes e_\mu)]$ (same for $e_{0,\lambda,\mu}$). Let

$$g_1 = g_1 \otimes \cdots \otimes g_{m_1} \in T^{m_1}(F[\mathbb{Z}_2]), \qquad g_2 = g_{m_1+1} \otimes \cdots \otimes g_n \in T^{m_2}(F[\mathbb{Z}_2])$$

and let M_1, M_2 be two monomials such that

$$M_1(x_1, \dots, x_{m_1}) \cdot M_2(x_{m_1+1}, \dots, x_n) \in F[S_{m_1} \times S_{m_2}] \subseteq F[S_n] \subseteq F[\mathbb{Z}_2 \sim S_n].$$

It easily follows by 2.4(1) that

$$\begin{aligned} &[(g_1 \otimes g_2) \otimes 1][M_1(x_1, \dots, x_{m_1}) \cdot M_2(x_{m+1}, \dots, x_n)] \\ &\quad = M_1\left(x_1^{g_1^{-1}}, \dots, x_{m_1}^{g_{m_1}^{-1}}\right) \cdot M_2\left(x_{m_1+1}^{g_{m_1+1}^{-1}}, \dots, x_n^{g_n^{-1}}\right). \end{aligned}$$

Replacing monomials by polynomials we now have

5.3. Corollary. *With the above notations,*

$$\begin{aligned} e_{0,\lambda,\mu} &= [f(m_1, m_2) \otimes 1] \cdot [e_{0,\lambda}(x_1, \dots, x_{m_1}) \cdot e_{0,\mu}(x_{m_1+1}, \dots, x_n)] \\ &= e_{0,\lambda}[x_1 + x_1^*, \dots, x_{m_1} + x_{m_1}^*] \cdot e_{0,\mu}[x_{m_1+1} - x_{m_1+1}^*, \dots, x_n - x_n^*]. \end{aligned}$$

5.4. Remark. The set $\{\gamma^{-1} e_{\lambda,\mu} \gamma \mid T(\lambda, \mu), \gamma \in \Gamma\}$ is a complete set of primitive idempotents in $I_{\lambda,\mu}$ (A.15). By (A.16), for any such $\gamma^{-1} e_{\lambda,\mu} \gamma$, there exists $\eta \in S_n$ such that $\gamma^{-1} e_{\lambda,\mu} \gamma = \eta^{-1} e_{0,\lambda,\mu} \eta$. Thus $\gamma^{-1} e_{\lambda,\mu} \gamma$ can now be realized in $V_n(x \mid *)$ by 5.3 and 2.5(1), (2).

5.5. We now follow [12, §2] and identify some of the variables x_i's in $e_{\lambda,\mu}(x)$. Note that $T^n(F[\mathbb{Z}_2])$ acts on any monomial of degree n – hence on homogeneous such polynomials – not necessarily multilinear.

If φ is the identification and $\varphi : x \to z$, then $\varphi : x^* \to z^*$ (3.3(a)). Thus φ commutes with $T^n(F[\mathbb{Z}_2]) \subseteq F[\mathbb{Z}_1 \sim S_n]$:

$$\varphi[(f(m_1,m_2)\otimes 1)(e_{0,\lambda}(x)\cdot e_{0,\mu}(x))] = (f(m_1,m_2)\otimes 1)\varphi(e_{0,\lambda}(x)\cdot e_{0,\mu}(x)).$$

As in [12, §2], rename the variables according to the tableaux $(T_0(\lambda), T_0(\mu))$, then identify: those in the ith row of $T_0(\lambda)$ are identified with y_i, those in the ith row of $T_0(\mu)$ with z_i. We thus obtain

5.6. Lemma. *Let λ' (μ') be the conjugate partition of λ (μ). Under the above $*$-substitution φ,*

$$\varphi(e_{0,\lambda,\mu}(x)) = d\left(\prod_j s_{\lambda_j'}[y_1+y_1^*,\ldots,y_{\lambda_j'}+y_{\lambda_j'}^*]\right)\left(\prod_l s_{\mu_l'}[z_1-z_1^*,\ldots,z_{\mu_l'}-z_{\mu_l'}^*]\right)$$

for some integer $d\neq 0$.

5.7. Remark. Since the $*$-codimensions of a $*$-P.I. ring are exponentially bounded, the rest of the results of [12] can immediately be generalized to such rings with involution. In particular, one can obtain explicit $*$-identities

$$(s_{l_1}^{k_1}[x+x^*])(s_{l_2}^{k_2}[x-x^*])$$

for such rings.

In fact, the whole body of results in this direction ([2], [12], [13], etc.) can now be generalized. Another possible generalization in that direction might be to G-P.I. rings.

We list below some of the theorems, with few hints as to their proofs.

Let $d_{t+1}[x_1,\ldots,x_{t+1}; y_1,\ldots,y_t]$ denote the Capelli polynomial:

$$d_{t+1}[x_1,\ldots,x_{t+1}; y_1,\ldots,y_t] = \sum_{\sigma\in S_{t+1}} \operatorname{sgn}(\sigma)x_{\sigma(1)}y_1x_{\sigma(2)}y_2\cdots y_tx_{\sigma(t+1)}.$$

5.8. Theorem [13, Theorem 2]. *Let R be an algebra with involution $*$ and let*

$$\chi_n(R\mid *) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

be its cocharacters: here $\chi_{\lambda,\mu}$ is the $\mathbb{Z}_2\sim S_n$ irreducible character that corresponds to (λ,μ) and $m_{\lambda,\mu}=m_{\lambda,\mu}(R\mid *)$ are the multiplicities.

(a) *R satisfies the $*$-Capelli identity*

$$d_{t+1}[x_1+x_1^*,\ldots,x_{t+1}+x_{t+1}^*; y_1,\ldots,y_t] = d_{t+1}[x+x^*; y]$$

iff

$$\chi_n(R\mid *) = \sum_{\substack{|\lambda|+|\mu|=n\\ h(\lambda)\le t}} m_{\lambda,\mu}\chi_{\lambda,\mu} \qquad (h(\lambda)=\lambda_1' \text{ is the height of } \lambda).$$

(b) *Similarly, R satisfies* $d_{u+1}[x-x^*;y]$ *iff*

$$\chi_n(R \mid *) = \sum_{\substack{|\lambda|+|\mu|=n \\ n(\mu)\le u}} m_{\lambda,\mu}\chi_{\lambda,\mu}.$$

(c) *R satisfies both* (a) *and* (b) *iff its* $*$*-cocharacters are 'contained' in a double strip!*

Hints for the proof. Follow the proof of the original ('ordinary') theorem [13, Theorem 2]. The three main ingredients in that proof are: the properties of $V_n(x)$ as a left and as a right FS_n module, and the 'Branching' rules in FS_n. These first two properties are generalized in 2.5(1), (2), while the corresponding branching theorem for $F[\mathbb{Z}_2 \sim S_n]$ is given here in A.19.

The rest of the proof now follows. $\square$

These same remarks imply

5.9. Theorem [2]. *Let R be as in* 5.8. *Then there exist* $k_1, l_1, k_2, l_2 \in \mathbb{N}$ *such that* $\chi_n(R \mid *)$ *'is contained' in the double hooks* $(H(k_1,l_1), H(k_2,l_2))$:

$$\chi_n(R,*) = \sum_{(\lambda,\mu)\in H_2(n)} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

where $H_2(n) = \{(\lambda,\mu) \mid |\lambda|+|\mu| = n,\ \lambda \in H(k_1,l_1),\ \mu \in H(k_2,l_2)\}$.

($H(k,l)$ is defined as the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ that satisfy $\lambda_{k+j} \le l$, $j = 1, 2, \ldots$). Similarly, the other results of [2] can be generalized!

5.10. Remarks. Hooks of Young diagrams were studied in [3]; applications to (ordinary) P.I. algebras were given in [4]. A generalization of the results of [3] to 'multihooks' was given in [14, §7]. The generalization of the results of [4] to rings with involution – and to G-P.I. rings – is yet to be done!

5.11. Conjecture. Let $R, \chi_n(R \mid *)$ as in 5.8. Then $f(n) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}(R \mid *)$ is polynomially bounded (as a function of n).

6. The matrix algebra F_k

Let F_k denote the $k \times k$ matrices, and let A^* be the transpose of $A \in F_k$: $* : A \to A^*$ is an involution! We now look closer at the $*$-identities of F_k.

6.1. Lemma. *Let* $t = \frac{1}{2}k(k+1)$, $u = \frac{1}{2}k(k-1)$ (*so* $t+u=k^2$), *then both t and u are minimal indices for which* F_k *satisfies the* $*$*-Capelli identities*

$$d_{t+1}[x+x^*;y] \quad \text{and} \quad d_{u-1}[x-x^*;y].$$

Proof. By a trivial dimension argument, F_k satisfies both these identities: the matrices $A+A^* \in F_k$ are symmetric, and their dimension is t. Similarly for u.

We prove the minimality of, say, t: order

$$\{(i,j) \mid i \le j\} = \{(i_1, j_1), \dots, (i_t, j_t)\},$$

then define:

$$\bar{x}_\nu = e_{i_\nu j_\nu} \qquad 1 \le \nu \le t,$$
$$\bar{y}_\nu = e_{j_\nu i_{\nu+1}}, \quad 1 \le \nu \le t-1,$$
$$\bar{y}_0 = e_{1 i_1} \quad \text{and} \quad \bar{y}_t = e_{j_t 1}.$$

Now evaluate

$$\bar{y}_0 d_t[\bar{x}_1 + \bar{x}_1^*, \dots, \bar{x}_t + \bar{x}_t^*; \bar{y}_1, \dots, \bar{y}_{t-1}] \bar{y}_t$$

by calculating that alternating sum over $\sigma \in S_t$; trivially, if $\sigma \neq 1$, its corresponding summand is zero! Hence

$$\bar{y}_0 \cdot d_t[\bar{x} + \bar{x}^*; \bar{y}] \cdot \bar{y}_t$$
$$= e_{1 i_1}(e_{i_1 j_1} + e_{j_1 i_1}) e_{j_1 i_2} \cdots (e_{i_t j_t} + e_{j_t i_t}) e_{j_t 1} = 2^k \cdot e_{11} \neq 0.$$

Similarly for u. $\square$

An immediate corollary of 5.8 and 6.1 is

6.2. Theorem. *Let $\chi_n(F_k \mid *)$ be the $*$-cocharacter of F_k, $t = \frac{1}{2}k(k+1)$ and $u = \frac{1}{2}k(k-1)$. Then*

$$\chi_n(F_k \mid *) = \sum_{\substack{|\lambda| + |\mu| = n \\ \lambda_1' \le t, \mu_1' \le u}} m_{\lambda,\mu} \cdot \chi_{\lambda,\mu} \qquad (\lambda_1' = h(\lambda) \textit{ is the height of } \lambda, \textit{ etc.})$$

Moreover, there exists $n = n(k)$ and partitions λ, μ, $|\lambda| + |\mu| = n$, satisfying $\lambda_1' = t$ and $\mu_1' = u$, for which the corresponding multiplicity $m_{\lambda,\mu} = m_{\lambda,\mu}(F_k \mid *)$ is nonzero.

6.3. Remarks [12, Theorem 3]. For the ordinary identities of F_k

$$\chi_n(F) = \sum_{\substack{\theta \vdash n \\ \theta_1' \le k^2}} m_\theta \cdot \chi_\theta.$$

Note that both $t, u \approx \frac{1}{2}k^2$ (and $t + u = k^2$) in 6.2. Thus, in a vague (!) sense, a single partition $\theta \vdash n$ in $\chi_n(F_k)$ is replaced, in $\chi_n(F_k \mid *)$, by pairs of partitions (λ, μ), $|\lambda| + |\mu| = n$, with $\approx$ half of the (possible) height of θ.

We also remark that at the moment, very little is known about the multiplicities $m_\lambda = m_\lambda(F_k)$ (in $\chi_n(F_k)$) if $k \ge 3$. A detailed study of the multiplicities $m_{\lambda,\mu}(F_k \mid *)$ might shed some light on these m_λ's.

We finally remark that if Conjecture 5.11 is true, it would imply – by asymptotic computations – that the two kinds of codimensions, $c_n(F_k \mid *)$ and $c_n(F_k)$, are very close to each other.

Appendix: $F[\mathbb{Z}_2 \sim S_n]$

The representation theory of $\mathbb{Z}_2 \sim S_n$ has been worked out by A. Young [17]. We shall now deduce that same theory, very easily, from the results of [14]. We give here a complete set of primitive idempotents for $I_{\lambda,\mu}$ that decompose it into a direct sum of minimal left ideals $J_{\lambda,\mu}$, in $F[\mathbb{Z}_2 \sim S_n]$. We also obtain a very explicit description of these one sided ideals $J_{\lambda,\mu} \subseteq I_{\lambda,\mu}$. We finally derive the branching rule for $F[\mathbb{Z}_2 \sim S_n]$.

All this can easily be generalized to the more general wreath products $A \sim S_n$.

A.1. Notations. Let $\mathbb{Z}_2 = \{1, g\}$, $g^2 = 1^2 = 1$, $A = F[\mathbb{Z}_2]$. Let $f_1 = \frac{1}{2}(1+g)$, $f_2 = \frac{1}{2}(1-g)$ in A, $A_i = Ff_i \cong F$, $i = 1, 2$, so $A = A_1 \otimes A_2$. We shall constantly refer to [14]. For a given (fixed) n we choose W with $\dim W \geq n$, so $A_i = X_i$ and $Z_i = X_i \otimes W \equiv f_i \otimes W \overset{\text{def}}{=} W_i$, $i = 1, 2$ [14, 5.2]. Let m be an integer, $\nu \vdash m$, T_ν a tableau of shape ν with corresponding idempotent $e_\nu \in FS_m$. We denote this by $\nu \leftrightarrow e_\nu$ (we shall later make the choice of T_ν more specific).

Let $n = m_1 + m_2$ and identify $F[S_{m_1} \times S_{m_2}] \equiv FS_{m_1} \otimes FS_{m_2}$. Let $\lambda \vdash m_1$, $\mu \vdash m_2$, $\lambda \leftrightarrow e_\lambda \in FS_{m_1}$, $\mu \leftrightarrow e_\mu \in FS_{m_2}$ so that $\langle \lambda, \mu \rangle \leftrightarrow e_\lambda \otimes e_\mu \in FS_{m_1} \otimes FS_{m_2}$. We also write $FS_{m_1} e_\lambda = J_\lambda$, $FS_{m_2} e_\mu = J_\mu$, the corresponding minimal left ideals.

A.2. Definition. With f_1, f_2 as in A.1, define

$$f(m_1, m_2) = T^{m_1}(f_1) \otimes T^{m_2}(f_2) \in T^n(A),$$

and denote

$$L_{\lambda,\mu} = f(m_1, m_2) \otimes (J_\lambda \otimes J_\mu) \subseteq A \sim S_n.$$

A.3. Notation. Let Λ (Γ) be a left (right) transversal of $S_{m_1} \times S_{m_2}$ in S_n:

$$S_n = \bigcup_{\tau \in \Lambda} \tau (S_{m_1} \times S_{m_2}) \qquad \left(S_n = \bigcup_{\gamma \in \Gamma} (S_{m_1} \times S_{m_2}) \gamma \right),$$

so that the coset-representative of $S_{m_1} \times S_{m_2}$ is $\tau = 1$. Also, write $s = \sum_{\tau \in \Lambda} \tau$ and denote

$$e_{\lambda,\mu} = f(m_1, m_2) \otimes (e_\lambda \otimes e_\mu), \qquad \bar{e}_{\lambda,\mu} = s \cdot e_{\lambda,\mu}.$$

A.4. Recall from [14]: $V = A \otimes W = W_1 \oplus W_2$, and $\varphi : A \sim S_n \to \mathrm{End}(T^n(V))$ is 1-1 ($\dim W \geq n$); by a slight abuse of notation we shall denote by φ also all the restrictions of φ. Recall also that

$$M_{\langle \lambda, \mu \rangle} = \varphi(e_\lambda \otimes e_\mu)(T^{m_1}(W_1) \otimes T^{m_2}(W_2)),$$

$$u_{A,W} = \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \qquad \text{(in this case)},$$

$$N_{\langle \lambda, \mu \rangle} = \mathrm{Hom}_{u_{A,W}}(M_{\langle \lambda, \mu \rangle}, T^n(V)),$$

and $\varphi^{-1}(N_{\langle\lambda,\mu\rangle})\overset{\text{def}}{=}J_{\lambda,\mu}$ is a minimal left ideal in $A\sim S_n$. The minimal two-sided ideal $I_{\lambda,\mu}\subseteq A\sim S_n$ is defined by $J_{\lambda,\mu}\subseteq I_{\lambda,\mu}$ (also, $I_{\lambda,\mu}=J_{\lambda,\mu}\cdot(A\sim S_n)$).

A.5. Note. It is well known that $\mathrm{Hom}_{\mathrm{GL}(W)}(\varphi(e_\lambda)T^{m_1}(W),T^{m_1}(W))=\varphi(FS_{m_1}e_\lambda)$ (and similarly for m_2 and e_μ). Now, $W_1=Ff_1\otimes W$ and we have

$$\begin{aligned}H_1 &\overset{\text{def}}{=}\mathrm{Hom}_{\mathrm{GL}(W_1)}(\varphi(e_\lambda)T^{m_1}(W_1),T^{m_1}(W_1))\\ &\equiv 1_{T^{m_1}(Ff_1)}\otimes\mathrm{Hom}_{\mathrm{GL}(W)}(\varphi(e_\lambda)T^{m_1}(W),T^{m_1}(W))\end{aligned}$$

(trivial). Since $\varphi(T^{m_1}(f_1))=1_{T^{m_1}(Ff_1)}$, we conclude that $H_1=\varphi(T^{m_1}(f_1)\otimes FS_{m_1}e_\lambda)$ (and similarly for m_2 and e_μ).

We now prove:

A.6. Theorem. *With the notations of* A.2, A.3 *and* A.4,

(a) $J_{\lambda,\mu}=(A\sim S_n)e_{\lambda,\mu}=(A\sim S_n)\bar{e}_{\lambda,\mu}$, *so both* $e_{\lambda,\mu},\bar{e}_{\lambda,\mu}$ *are primitive idempotents.*

(b) $J_{\lambda,\mu}=\bigoplus_{\tau\in\Lambda}\tau\cdot L_{\lambda,\mu}$.

Proof. Let $P=T^{m_1}(W_1)\otimes T^{m_2}(W_2)$. It easily follows from (the proofs of) [14, 5.6, 5.7 and 5.8] and from A.5 that

$$\begin{aligned}\varphi(J_{\lambda,\mu}) &= N_{\langle\lambda,\mu\rangle}=\bigoplus_{\tau\in\Lambda}\varphi(\tau)\mathrm{Hom}_{u_{A,W}}(\varphi(e_\lambda\otimes e_\mu)P,P)\\ &=\varphi(s)(\mathrm{Hom}_{\mathrm{GL}(W_1)}(\varphi(e_\lambda)T^{m_1}(W_1),T^{m_1}(W_1)))\\ &\quad\otimes\mathrm{Hom}_{\mathrm{GL}(W_2)}(\varphi(e_\mu)T^{m_2}(W_2,T^{m_2}(W_2)))\\ &=\varphi(s)(\varphi(f(m_1,m_2)\otimes(FS_{m_1}e_\lambda\otimes FS_{m_2}e_\mu)))\\ &=\varphi[s(f(m_1,m_2)\otimes(J_\lambda\otimes J_\mu))].\end{aligned}$$

Here $s=\sum_{\tau\in\Lambda}\tau$ and $f(m_1,m_2)=T^{m_1}(f_1)\otimes T^{m_2}(f_2)$. Since $\dim W\ge n,\varphi$ is 1-1, so we conclude that $J_{\lambda,\mu}=\oplus_{\tau\in\Lambda}\tau\ L_{\lambda,\mu}=s[f(m_1,m_2)\otimes(J_\lambda\otimes J_\mu)]$, which proves (b), and also implies that $e_{\lambda,\mu},\bar{e}_{\lambda,\mu}\in J_{\lambda,\mu}$. To prove (a) we show that $e^2_{\lambda,\mu}=e_{\lambda,\mu}$ and $\bar{e}^2_{\lambda,\mu}=\bar{e}_{\lambda,\mu}$. But this is a trivial consequence of the following

A.7. Lemma. *Let* $n=m_1+m_2$, $\lambda\vdash m_1$, $\mu\vdash m_2$, $\lambda\leftrightarrow e_\lambda$, $e'_\lambda\in FS_{m_1}$, $\mu\leftrightarrow e_\mu$, $e'_\mu\in FS_{m_2}$ *as in* A.1, *and let* $\theta\in S_n$. *Then*

$$\begin{aligned}&[f(m_1,m_2)\otimes(e_\lambda\otimes e_\mu)]\,\theta[f(m_1m_2)\otimes(e'_\lambda\otimes e'_\mu)]\\ &\quad=\begin{cases}0 & \text{if } \theta\notin S_{m_1}\times S_{m_2},\\ f(m_1,m_2)\otimes(e_\lambda\theta_1e_\mu\otimes e'_\lambda\theta_2e'_\mu) & \text{if } \theta=(\theta_1,\theta_2)\in S_{m_1}\times S_{m_2}.\end{cases}\end{aligned}$$

Proof. Note that

$$Q=[f(m_1,m_2)\otimes(e_\lambda\otimes e_\mu)]\,\theta=f(m_1,m_2)\otimes[(e_\lambda\otimes e_\mu)\theta]$$

So, if $\theta \notin S_{m_1} \times S_{m_2}$, then

$$Q = \sum_{\sigma \notin S_{m_1} \times S_{m_2}} \alpha_\sigma \cdot f(m_1, m_2) \otimes \sigma \qquad (\alpha_\sigma \in F).$$

Now, $\sigma[f(m_1,m_2) \otimes (e'_\lambda \otimes e'_\mu)] = \sigma(f(m_1,m_2)) \otimes \sigma(e'_\lambda \otimes e'_\mu)$ and since $\sigma \notin S_{m_1} \times S_{m_2}$, $f(m_1,m_2) \cdot \sigma(f(m_1,m_2)) = 0$ $(f_1 \cdot f_2 = 0)$, which clearly implies the first part. The proof of the second part is similar and is based on the (obvious) fact that if $\theta \in S_{m_1} \times S_{m_2}$, then $\theta(f(m_1,m_2)) = f(m_1,m_2) = f^2(m_1,m_2)$.

This completes the proof of the Lemma, which clearly implies that $e^2_{\lambda,\mu} = e_{\lambda,\mu}$ and $\bar{e}^2_{\lambda,\mu} = \bar{e}_{\lambda,\mu}$; thus completing A.6. □

To complete our investigation of $I_{\lambda,\mu}$ we now give a complete system of primitive idempotents that decompose $I_{\lambda,\mu}$ as a direct sum of minimal left ideals. These idempotents, in general, are not orthogonal.

First, from A.7 we deduce

A.8. Corollary. *Let $\lambda \leftrightarrow e_\lambda, e'_\lambda$, $\mu \leftrightarrow e_\mu, e'_\mu$ and $e_{\lambda,\mu}, e'_{\lambda,\mu}$ as in* A.3. *If $(e_\lambda \otimes e_\mu)(e'_\lambda \otimes e'_\mu) = 0$, then $e_{\lambda,\mu} \cdot e'_{\lambda,\mu} = 0$ (and $\bar{e}_{\lambda,\mu} \cdot \bar{e}_{\lambda,\mu} = 0$).* (Obvious.)

A.9. Notation. Let $n = m_1 + m_2$, $\lambda \vdash m_1$, $\mu \vdash m_2$ and denote

$$T(\lambda, \mu) = \left\{ (T_\lambda, T_\mu) \ \begin{matrix} T_\lambda \text{ is standard of shape } \lambda \\ T_\mu \text{ is standard of shape } \mu \end{matrix} \right\}.$$

Each $(T_\lambda, T_\mu) \in T(\lambda, \mu)$ defines an $e_{\lambda,\mu}$, as in A.3, hence the corresponding minimal left ideal $(A \sim S_n)e_{\lambda,\mu} = J_{\lambda,\mu}$. We denote

$$k(\lambda, \mu) = \sum_{T(\lambda,\mu)} (A \sim S_n) e_{\lambda,\mu}.$$

A.10. Note. Order $\{T_\lambda\}$ lexicographically, let $T_\lambda < T'_\lambda$ and let $T_\lambda \leftrightarrow e_\lambda$, $T'_\lambda \leftrightarrow e'_\lambda$. Then it is well known that $e_\lambda \cdot e'_\lambda = 0$: the set $\{e_\lambda \mid e_\lambda \leftrightarrow T_\lambda\}$ is 'one-sided orthogonal'. Same for $\{e_\mu \mid e_\mu \leftrightarrow T_\mu\}$, and by a corresponding lexicographic order of $T(\lambda, \mu)$, $\{e_{\lambda,\mu} \mid e_{\lambda,\mu} \leftrightarrow (T_\lambda, T_\mu) \in T(\lambda, \mu)\}$ is also one-sided orthogonal. By A.7, if $e_{\lambda,\mu}, e'_{\lambda,\mu}$ are as in A.7 and $\gamma_1 \neq \gamma_2 \in \Gamma$, then $(\gamma_1^{-1} e_{\lambda,\mu} \gamma_1)(\gamma_2^{-1} e'_{\lambda,\mu} \gamma_2) = 0$, since $\gamma_1 \gamma_2^{-1} \notin S_{m_1} \times S_{m_2}$. Thus, by an appropriate order, $\{\gamma^{-1} e_{\lambda,\mu} \gamma \mid e_{\lambda,\mu} \leftrightarrow (T_\lambda, T_\mu) \in T(\lambda, \mu), \gamma \in \Gamma\}$ is 'one-sided orthogonal'. By a standard argument, this yields

A.11. Remark. Let $K(\lambda, \mu)$ be as in A.9. Then

$$K(\lambda, \mu) = \bigoplus_{T(\lambda,\mu)} (A \sim S_n) e_{\lambda,\mu} = \bigoplus_{T(\lambda,\mu)} J_{\lambda,\mu}.$$

Thus $\dim K(\lambda, \mu) = \binom{n}{m_1} d^2_\lambda \cdot d^2_\mu$.

A.12. Note. It is well known that if $\{T_\lambda\}$ are all the standard tableaux of shape λ, then $\sum_{\{T_\lambda\}} FS_{m_1} e_\lambda = \sum_{\{T_\lambda\}} J_\lambda = I_\lambda$, the minimal two-sided ideal $I_\lambda \subseteq FS_{m_1}$. Same for μ. This implies

A.13. Lemma. $K(\lambda,\mu)=\bigoplus_{\tau\in\Lambda}\tau[f(m_1,m_2)\otimes(I_\lambda\otimes I_\mu)]$.

Proof. By A.6(b), $J_{\lambda,\mu}\supseteq f(m_1,m_2)\otimes(J_\lambda\otimes J_\mu)$, hence

$$K(\lambda,\mu)=\bigoplus_{T(\lambda,\mu)} J_{\lambda,\mu}$$
$$\supseteq f(m_1,m_2)\otimes\left(\sum_{T(\lambda,\mu)}(J_\lambda\otimes J_\mu)\right)=f(m_1,m_2)\otimes(I_\lambda\otimes I_\mu).$$

Since $K(\lambda,\mu)$ is a left ideal, $K(\lambda,\mu)\supseteq\sum_{\tau\in\Lambda}\tau[f(m_1,m_2)\otimes(I_\lambda\otimes I_\mu)]$. The r.h.s is clearly a direct sum and therefore its dimension is (also) $\binom{n}{m_1}d_\lambda^2d_\mu^2$; hence, by A.11, it is equal to the l.h.s. □

A.14. Lemma. *With the above notations we have*

$$[f(m_1,m_2)\otimes(I_\lambda\otimes I_\mu)]\cdot[T^n(A)\otimes(FS_{m_1}\otimes FS_{m_2})]=f(m_1,m_2)\otimes(I_\lambda\otimes I_\mu).$$

Proof. Clearly, l.h.s. $\supseteq$ r.h.s. Since $T^n(A)$ is F-spanned by the elements

$$f(i)=f_{i_1}\otimes\cdots\otimes f_{i_n},\quad i_j\in\{1,2\},$$

the proof now easily follows from the following observation:
If $\sigma\in S_{m_1}\times S_{m_2}$, then

$$(f(m_1,m_2)\otimes\sigma)\cdot f(i)=\begin{cases}0 & \text{if } f(m_1,m_2)\neq f(i),\\ f(m_1,m_2)\otimes\sigma & \text{if } f(i)=f(m_1,m_2).\end{cases}\qquad\square$$

A.15. Theorem. *With the notations of* A.3, $I_{\lambda,\mu}=\bigoplus_{\Gamma,T(\lambda,\mu)}(A\sim S_n)(\gamma^{-1}e_{\lambda,\mu}\gamma)$, *i.e.* $\{\gamma^{-1}e_{\lambda,\mu}\gamma\}$ *is a complete set of primitive idempotents for* $I_{\lambda,\mu}$ *(each* $(A\sim S_n)(\gamma^{-1}e_{\lambda,\mu}\gamma)=(A\sim S_n)e_{\lambda,\mu}\gamma$ *is a minimal left ideal in* $A\sim S_n$*).*

Proof. Since $I_{\lambda,\mu}$ is the minimal two-sided ideal $\supseteq J_{\lambda,\mu}$, hence $I_{\lambda,\mu}=K(\lambda,\mu)\cdot(A\sim S_n)$. Clearly $A\sim S_n=\bigoplus_{\gamma\in\Gamma}[T^n(A)\otimes(FS_{m_1}\otimes FS_{m_2})]\gamma$, hence, by A.14, $I_{\lambda,\mu}=\sum_{\gamma\in\Gamma}K(\lambda,\mu)\gamma$. Now, $\dim I_{\lambda,\mu}=\binom{n}{m_1}^2d_\lambda^2d_\mu^2$, $|\Gamma|=\binom{n}{m_1}$ and $\dim K(\lambda,\mu)=\binom{n}{m_1}d_\lambda^2d_\mu^2$ (A.11), therefore $I_{\lambda,\mu}=\bigoplus_{\gamma\in\Gamma}K(\lambda,\mu)\gamma$. The theorem now follows from A.11. □

A.16. Remark. Let $e_1=\gamma_1^{-1}e_{\lambda,\mu}\gamma_1$, $e_2=\gamma_2^{-1}e'_{\lambda,\mu}\gamma_2$ be two idempotents as in A.15. Then there exists $\eta\in S_n$ such that $e_2=\eta^{-1}e_1\eta$.

Proof. This is well known in FS_n: there exist $\theta_i\in S_{m_i}$, $i=1,2$, such that

$$e'_\lambda=\theta_1^{-1}e_\lambda\theta_1\quad\text{and}\quad e'_\mu=\theta_2^{-1}e_\mu\theta_2.$$

Since $\theta=\theta_1\theta_2$ commutes with $f(m_1,m_2)$, hence $e'_{\lambda,\mu}=\theta^{-1}e_{\lambda,\mu}\theta$, so $e_2=\eta^{-1}e_1\eta$ where $\eta=\gamma_1{}^{-1}\theta\gamma_2$. □

A.17. Right ideals. We now decompose $I_{\lambda,\mu}$ into minimal right ideals.

Define $\varphi : A \sim S_n \to A \sim S_n$ by $a = a_1 \otimes \cdots \otimes a_n$, $\sigma \in S_n$, $\varphi(a \otimes \sigma) = \sigma^{-1}(a) \otimes \sigma^{-1}$, and extend linearly to $A \sim S_n$. Check that $\varphi((a \otimes \sigma) \cdot (b \otimes \tau)) = \varphi(b \otimes \tau) \cdot \varphi(a \otimes \sigma)$. Thus φ is an anti-isomorphism of $A \sim S_n$ with itself.

Clearly, $\varphi(e_\lambda \otimes e_\mu) = e_\lambda \otimes e_\mu$ and $\varphi(f(m_1, m_2)) = f(m_1, m_2)$, hence $\varphi(e_{\lambda,\mu}) = e_{\lambda,\mu}$. Thus $\varphi(J_{\lambda,\mu}) = \varphi((A \sim S_n) e_{\lambda,\mu}) = e_{\lambda,\mu}(A \sim S_n)$ is the corresponding minimal right ideal. The decomposition of $I_{\lambda,\mu}$ into such ideals is now clear.

Branching in $\mathbb{Z}_2 \sim S_n$

The embedding of $A \sim S_n$ into $A \sim S_{n+1}$ can be done in many ways, and we choose a natural one: S_n embeds naturally into S_{n+1} ($S_n = \{\sigma \in S_{n+1} \mid \sigma(n+1) = n+1\}$). Identify now $a_1 \otimes \cdots \otimes a_n \otimes \sigma \in A \sim S_n$ with $a_1 \otimes \cdots \otimes a_n \otimes \sigma \equiv a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \sigma \in A \sim S_{n+1}$, to have $A \sim S_n \subseteq A \sim S_{n+1}$.

Given $J_{\lambda,\mu} \subseteq A \sim S_n$ as before, we shall give its branching in $A \sim S_{n+1}$ by calculating $(A \sim S_{n+1}) J_{\lambda,\mu}$ as a sum of irreducibles in $A \sim S_{n+1}$.

A.18. Note. Let $R \supseteq S$ be finite-dimensional F-algebras with $1 = 1_R = 1_S$ such that R is a free right S module, and let $J \subset S$ be a left ideal. Then $RJ \cong R \otimes_S J$ (via $r \otimes j \to r \cdot j$). In particular,

$$(A \sim S_{n+1}) \cdot J_{\lambda,\mu} \cong (A \sim S_{n+1}) \otimes_{A \sim S_n} J_{\lambda,\mu}.$$

A.19. Notation. Let $\lambda \vdash m_1$ and identify a partition with its Young diagram. Then denote λ^+ = all the diagrams obtained from λ by adding one cell. Similarly for μ^+, where $\mu \vdash m_2$.

A.20. Branching in $FS_m \to FS_{m+1}$ is well known: If J_λ is a minimal left ideal in FS_m, then $FS_{m+1} \cdot J_\lambda \cong \bigoplus_{\lambda' \in \lambda^+} J_{\lambda'}$. We can now prove

A.21. The Branching theorem. *With the above notations,*

$$(A \sim S_{n+1}) J_{\lambda,\mu} \cong \left(\bigoplus_{\lambda' \in \lambda^+} J_{\lambda',\mu} \right) \oplus \left(\bigoplus_{\mu' \in \mu^+} J_{\lambda,\mu'} \right).$$

Proof. Since

$$T^{m_1}(f_1) \otimes T^{m_2}(f_2) \otimes 1 \otimes (e_\lambda \otimes e_\mu) \equiv T^{m_1}(f_1) \otimes T^{m_2}(f_2) \otimes (e_\lambda \otimes e_\mu) \in J_{\lambda,\mu}$$

we have $T^{m_1}(f_1) \otimes T^{m_2}(f_2) \otimes f_i \otimes (e_\lambda \otimes e_\mu) \in (A \sim S_{n+1}) J_{\lambda,\mu}$ for both $i = 1, 2$.

Case 1: $i = 2$, so $f(m_1, m_2+1) \otimes (e_\lambda \otimes e_\mu) \in (A \sim S_{n+1}) J_{\lambda,\mu}$. Since $f(m_1, m_2+1)$ commutes with $S_{m_1} \times S_{m_2+1}$,

$$\begin{aligned}(A \sim S_{n+1}) J_{\lambda,\mu} &\supseteq f(m_1, m_2+1) \otimes (J_\lambda \otimes FS_{m_2+1} e_\mu) \\ &\cong \bigoplus_{\mu' \in \mu^+} [f(m_1, m_2+1) \otimes (J_\lambda \otimes J'_\mu)].\end{aligned}$$

It follows from A.6 that

$$(A \sim S_{n+1}) \cdot J_{\lambda,\mu} \supseteq \sum_{\mu' \in \mu^+} J_{\lambda,\mu'}.$$

Moreover, $\sum_{\mu' \in \mu^+} J_{\lambda,\mu} = \bigoplus_{\mu' \in \mu^+} J_{\lambda,\mu}$ since these irreducibles are pairwise non-isomorphic.

Case 2: $I=1$. Similarly, let $\tilde{f}(m_1+1, m_2) = T^{m_1}(f_1) \otimes T^{m_2}(f_2) \otimes f_1$, and let

$$(S_{m_1+1} \times S_{m_2})^{\sim} = S_{m_1+1}(1,2,\dots,m_1,n+1) \times S_{m_2}(m_1+1,\dots,n).$$

Then $(S_{m_1+1} \times S_{m_2})^{\sim}$ and $\tilde{f}(m_1+1, m_2)$ commute. By exactly the same arguments as above. $(A \sim S_{n+1}) J_{\lambda,\mu} \supseteq \bigoplus_{\lambda' \in \lambda^+} \tilde{J}_{\lambda',\mu}$ where $\tilde{J}_{\lambda',\mu} \cong J_{\lambda',\mu}$. Since all these irreducibles are pairwise non-isomorphic,

$$(A \sim S_{n+1}) J_{\lambda,\mu} \supseteq \left(\bigoplus_{\lambda' \in \lambda^+} \tilde{J}_{\lambda',\mu} \right) \oplus \left(\bigoplus_{\mu' \in \mu^+} J_{\lambda,\mu'} \right).$$

Calculate dimensions:

$$\dim \tilde{J}_{\lambda',\mu} = \binom{n+1}{m_1+1} d_\lambda d_\mu, \qquad \dim J_{\lambda,\mu'} = \binom{n+1}{m_2+1} d_\lambda d_{\mu'},$$

so the dimension of the r.h.s is

$$\begin{aligned} &\sum_{\lambda' \in \lambda^+} \binom{n+1}{m_1+1} d_{\lambda'} d_\mu + \sum_{\mu' \in \mu^+} \binom{n+1}{m_2+1} d_\lambda d_{\mu'} \\ &= \frac{(n+1)!}{(m_1+1)!\, m_2!} d_\mu \cdot \underbrace{\left(\sum_{\lambda' \in \lambda^+} d_{\lambda'} \right)}_{=(m_1+1)d_\lambda} + \frac{(n+1)!}{m_1!\,(m_2+1)!} d_\lambda \cdot \underbrace{\left(\sum_{\mu' \in \mu^+} d_{\mu'} \right)}_{=(m_2+1)d_\mu} \\ &= 2(n+1) \binom{n}{m_1} d_\lambda d_\mu = 2(n+1) d_{\lambda,\mu} \end{aligned}$$

which obviously is the dimension of the l.h.s. Thus l.h.s = r.h.s, so

$$(A \sim S_{n+1}) J_{\lambda,\mu} \cong \left(\bigoplus_{\lambda' \in \lambda^+} J_{\lambda',\mu} \right) \oplus \left(\bigoplus_{\mu' \in \mu^+} J_{\lambda,\mu'} \right). \qquad \square$$

References

[1] S.A. Amitsur, Identities in rings with involution, Israel J. Math. 7 (1969) 63–68.

[2] S.A. Amitsur and A. Regev, P.I. algebras and their cocharacters, J. Algebra 78(1) (1980) 248–254.

[3] A. Berele and A. Regev, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, Advances in Math., to appear.

[4] A. Berele and A. Regev, Applications of hook Young diagrams to P.I. algebras, J. Algebra 82(2) (1982) 559–567.

[5] I.N. Herstein, Rings with Involution (Univ. of Chicago Press, Chicago, 1976).

[6] G. James and A. Kerber, The Representation Theory of the Symmetric Group, Encyclopedia of Math. and its Applications (Addison-Wesley, Reading, MA, 1981).
[7] V.K. Kharchenco, Identities involving automorphisms, Siberskii Matem. Zhurnal 17 (1976) 446-467 (Russian). (English transl. December 1976, 349-363.)
[8] I.G. MacDonald, Polynomial functors and wreath products, J. Pure Appl. Algebra 18 (1980) 173-204.
[9] S. Montgomery, Fixed rings of automorphism groups of associative rings. Lecture Notes in Math. 818 (Springer, Berlin, 1980).
[10] A. Regev, Existence of identities in $A\otimes B$, Isreal J. Math. 2(2) (1972) 131-152.
[11] A. Regev, The T-ideal generated by the standard identity $S_3[x_1,x_2,x_3]$, Israel J. Math. 26(2) (1977) 105-125.
[12] A. Regev, The representations of S_n and explicit identities for P.I. algebras, J. Algebra 51(1) (1978) 25-40.
[13] A. Regev, Algebras satisfying a Capelli identity, Israel J. Math. 33(2) (1979) 149-154.
[14] A. Regev, The representations of wreath products via double centralizing theorems, Preprint.
[15] L. Rowen, Polynomial Identities in Ring Theory (Academic Press, New York, 1980).
[16] W. Specht, Eine Verallgemeinerung der symmetrischen Gruppe, Schriften Math. Seminar Berlin 1 (1932) 1-32.
[17] A. Young, QSA V, Proc. London Math. Soc. (2) 31 (1930) 273-288.